TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

Pendulum masses
Vibration, torsional

No. 1035

CONTROL OF TORSIONAL VIBRATIONS BY PENDULUM MASSES

By Albert Stieglitz

Jahrbuch 1938 der Deutschen Luftfahrtforschung

Washington
November 1942

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Various versions of pendulum masses have been developed abroad within the past few years by means of which resonant vibrations of rotating shafts can be eliminated at a given tuning. They are already successfully employed on radial engines in the form of pendulous counterweights. Compared with the commonly known torsional vibration dampers, the pendulum masses have the advantage of being structurally very simple, requiring no internal damping and being capable of completely eliminating certain vibrations.

Unexplained, so far, remains the problem of behavior of pendulum masses in other critical zones to which they are not tuned, their dynamic behavior at some tuning other than in resonance, and their effect within a compound vibration system and at simultaneous application of several differently tuned pendulous masses.

These problems are analyzed in the present report. The results constitute an enlargement of the scope of application of pendulum masses, especially for in-line engines. Among other things it is found that the natural frequency of a system can be raised by means of a correspondingly tuned pendulum mass. The formulas necessary for the design of any practical version are developed, and a pendulum mass having two different natural frequencies simultaneously is described.

I. INTRODUCTION

The present investigation deals with the torsional vibrations of rotating shafts under the effect of pendulum masses, which have been added for the purpose of eliminating the dangers of resonant vibrations. The pendulously linked masses are subjected to the centrifugal forces of

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*"Beeinflussung von Drehschwingungen durch pendelnde Massen." Jahrbuch 1938 der Deutschen Luftfahrtforschung, pp. II 164-178.

the rotary motion which exert restoring forces on the vibratory motion of the pendulum masses.

Both Taylor's (reference 1) and Salomon's (reference 2) system of pendulum masses are based on the resonance principle in the same manner as the conventional resonance dampers. But, while in the latter the mass is coupled by springs or gravity with the system and so attains a certain constant natural frequency, the natural frequency of the pendulum mass restored by the centrifugal forces varies with the rotative speed, hence changes its dynamic behavior fundamentally.

Resonant dampers with constant natural frequency remove the vibrations in one critical speed, but permit the creation of a new critical zone above and below this speed, the damping of which requires an internal damping. Resonant dampers have been known for a long time. As early as 1895, Pollak (reference 3, p. 911) employed a flywheel disk fastened to a shaft by means of rubber bushings of a specific size for the purpose of reducing the shaft vibrations, and by so doing, undoubtedly, applied the resonance principle. In 1908 Schüler's resonance principle (reference 3, p. 845) had become known and accepted. In Frahm's antirolling tank of 1911 (reference 4) a water mass coupled with the ship by the centrifugal force is tuned to the same natural frequency as that of the ship. In the same year (1911), Kutzbach (reference 3, pp. 451 and 703) developed the damper shown in figure 1 on which a fluid mass in the U-shaped channel of a flywheel is restored by centrifugal forces. This device itself represents a forerunner of the pendulum masses of Taylor and Salomon. However, the dynamic relations were not explored in detail at that time, nor was the method further followed up, probably because there were no demands for such dampers then.

The pendulum mass is a means of eliminating dangerous vibrations, which, by virtue of its effectiveness and simplicity surpasses any other known damping device, and merits much greater attention. While, for instance, with one of the known dampers (resonant damper, friction damper) the vibrations are, at best, reduced to the static deflection corresponding to the exciting force, the vibrations of a one-mass system can be completely eliminated by one or more pendulum masses. The shaft then revolves like a rigid shaft without being alternatingly stressed, notwithstanding the alternating forces acting on it. The pendulum masses are not subject to wear in operation nor to variation in tuning in addition to being insusceptible to displacements of the natural frequencies of the system.

This principle has been utilized successfully in a device employed on the U. S. Wright-Cyclone radial aircraft engines. The already existing counterweight serves as pendulum mass. A Hispano-Suiza radial engine and a Peugeot automobile engine in France are also said to have been fitted with pendulum masses. Further applications of pendulum masses to other automobile engines or stationary plants are not known. The reason for this lies in their comparative newness and insufficient knowledge of their mode of operation. Apart from that, there is the matter of patents.

Taylor's studies extend to the case of a two-mass system with a simple pendulum mass. He deduced the natural frequency of the pendulum mass and the resultant tuning at a certain mode of excitation. But his premise for the general motion of the pendulous mass is wrong. The result for the general case is not discussed further. Salomon used a flywheel equipped with pendulum rollers. The general formula is developed only for the case of resonance of the pendulous roller. The natural frequencies for several forms of pendulous masses and their deflections in resonance are calculated. Both Taylor and Salomon treat only the case of resonance of the pendulous mass, but fail to touch upon the subject of how this mass really acts in other critical zones.

In the following an attempt is made to explain, in a very general fashion, the phenomenon accompanying the motions of pendulum masses. The study is extended to include any compound-mass system. Besides the resonant tuning of the pendulum masses other cases of tuning are dealt with, which, as will be seen, are often much more of practical importance. The different types of pendulum masses are analyzed and the necessary design data indicated.

II. PROCESS OF INVESTIGATION

The investigation was carried on with very simple equipment. Only small vibratory deflections φ are presumed, at which it is always possible to put $\sin \varphi = \varphi$, $\cos \varphi = 1$. The error involved is practically insignificant in most cases, when it is considered, for instance, that the natural frequency of a gravity pendulum at deflections up to $\pm 22^{\circ}$ diverges only about 1 percent from the value for small deflections.

The system subjected to torsional vibrations, performs a uniform rotary motion together with a superimposed vibrating motion. When applying the centrifugal forces at the individual pendulous masses, the rotary motion can for the present be disregarded. The attendant Coriolis component of acceleration due to the concomitant vibrating motion is negligible, since it is small at small deflections compared with the centrifugal forces of higher order. Then, if the mass forces of the oscillating motion are applied to all the masses the vibratory motion itself can be disregarded and the system is completely at rest. The task is thus reduced to a static problem and merely involves bringing all forces in the proper phase into equilibrium at the different points, so that all the dynamic relations can be secured. This simplifies the study considerably and makes for a clear representation of the processes. This method has been previously employed by the author in his article titled: "Torsional Vibrations in In-Line Engines" (reference 5), which also contains a detailed description of the method.

In conformity with the available data on pendulum masses, the case of a system with one degree of freedom and a simple pendulum mass is treated first. The forced, damped vibrations of this system are analyzed for different tuning of pendulum mass and different modes of excitation, and subsequently the general case of a pendulum mass attached at some point of a compound-mass system. Practically every important case of pendulum mass is mathematically worked out. Lastly, a pendulum mass with two natural frequencies comprising two different modes of torsional vibrations simultaneously is discussed.

The excitation is assumed constant over the entire rotative speed range. If an excitation increasing with the square of the rotative speed is to be used as basis, such as approximately occurs on an Otto-type of aircraft engine, all the deflections obtained are merely changed proportionally $(\omega/\omega_e)^2$. This is secondary for the operation of the pendulum masses. The motion of the pendulum mass is, in addition, assumed to be free from friction.

III. ONE LINK VIBRATION SYSTEM WITH ONE PENDULUM MASS

It is to be noted beforehand, that the subject treated in this chapter is also contained in the general solution of the subsequent chapter; nevertheless, this simple case is to be treated by itself as introduction and continuation of the already existing research data.

Visualize a vibrating system (fig. 2) consisting of a flywheel mass of moment of inertia Θ and a shaft with spring constant C, having a point-pendulum mass m_0 with length of pendulum s, linked at distance r from the axis of rotation. Then visualize the system built in at A into an infinitely large flywheel and rotating with it at an angular speed Ω. The flywheel mass Θ can be replaced, as indicated in figure 3, by a mass m at distance $(r + s)$ from the axis of rotation, whereby $\Theta = m(r + s)^2$. Then the longitudinal vibrations of the masses on the circumference of the circle with radius $(r + s)$ can be investigated for small deflections instead of the torsional vibrations.

Considering, next, the mass m of the system as nonvibratory and applying the centrifugal force $F = m_0(r+s)\Omega^2$ on the pendulum mass m_0, the mass m_0 is restored by the force $F(\epsilon - \psi)$ after a vibratory deflection ψ, according to figure 3. An identical force is applied at m as counterforce in opposite direction. (See appendix 1.) Then

$$\psi(r + s) = \epsilon s$$

$$\epsilon - \psi = \frac{r}{s}\psi, \text{ hence}$$

$$F(\epsilon - \psi) = m_0(r + s)\Omega^2 \frac{r}{s}\psi = c_0\Omega^2 b,$$

where $b = \psi(r + s)$ is the longitudinal deflection of mass m_0 corresponding to the angular deflection ψ. The restoration of m_0 relative to m therefore takes place with a spring stiffness $c_0\Omega^2$ which varies proportional to the square of the rotative speed, whereby $c_0 = m_0 \frac{r}{s}$. Then the original system of figure 2 can be replaced by the simple nonrotating two-mass system of figure 4, that differs from a common two-mass system merely by the spring stiffness $c_0\Omega^2$ increasing with the speed of rotation. By spring constant c or $c_0 \Omega^2$ is meant the stiffness, that is, the force required for the deflection 1, not the spring action which forms the reciprocal value of it.

The natural frequency ω_{e0} of the pendulum mass in comparison with the static mass of the system then is

$$\omega_{e0}^2 = \frac{c_0 \Omega^2}{m_0} = \frac{r}{s} \Omega^2$$

according to which the natural frequency is proportional to the rotative speed. The ratio ω_{e0}/Ω is designated with q and termed the tuning of the pendulum mass, since it governs the behavior of the pendulum mass. Thus the tuning of the pendulum mass is, in the present case

$$\boxed{q^2 = \frac{r}{s}} \tag{1}$$

Now the system of figure 4 is to perform forced damped vibrations under the effect of a harmonic excitation $P \sin \omega t$ acting on m. The amplitude of the mass m is a, that of mass m_0 is a_0, and the relative deflection is b. The system has natural damping; the damping force acting conversely to the speed on m is put at ka, that is, independent of the frequency, as most closely approached by a material damping. Then the work of hysteresis is proportional to the square of the material stress.

Applying all the forces, including the mass forces in the correct phase, affords the force diagram of figure 5. The excitation P is to have the phase β relative to the oscillating deflection a. The formation of this diagram can be envisaged as the actual vibratory motion being superposed by an exactly identical vibratory motion shifted by a difference of phase of 90° relative to the former in respect to space and time. Herewith all vibratory motions become simple circular motions with rotational speed ω, the forces assume constant values.

The equilibrium conditions of all forces yield the three equations:

$$\left.\begin{aligned} m_0 a_0 \omega^2 &= c_0 \Omega^2 b \\ c a &= P \cos\beta + c_0 \Omega^2 b + m a \omega^2 \\ k a &= P \sin\beta \end{aligned}\right\} \quad (2)$$

which suffice for the determination of the three unknowns, a, a_0, and β, whereby $b = a_0 - a$. With

$p = \omega/\Omega$ mode of excitation (that is, $= 1, 2, 3, 4, \ldots$ for two cycle)($= \frac{1}{2}, 1, 1\frac{1}{2} \ldots$ for four cycle)

$a_{st} = P/c$ static deflection of the mass of the system

$z = k/c$ damping, that is, the ratio of damping force to spring force

$V = a/a_{st}$ magnification

$\omega_e^2 = c/m$ natural frequency of the system

the foregoing equation (cf. appendix 2) affords the following solution in dimensionless form:

The magnification is

$$V = \frac{a}{a_{st}} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_e}\right)^2\left(1 + \frac{m_0}{m}\,\frac{1}{1 - \left(\frac{p}{q}\right)^2}\right)\right]^2 + z^2}}$$

or

$$\boxed{V = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_e'}\right)^2\right]^2 + z^2}}} \quad (3)$$

where

$$\boxed{\left(\frac{\omega_e'}{\omega_e}\right)^2 = \frac{1}{1 + \frac{m_0}{m}\,\frac{1}{1 - (p/q)^2}}} \quad (4)$$

The relative deflections of the pendulum mass are according to appendix 2:

$$\boxed{\frac{b}{a} = \frac{1}{(q/p)^2 - 1}} \tag{5}$$

Putting $m_o = 0$, that is, omitting the pendulum mass, leaves the conventional resonance curve of the common one-mass system

$$V = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_e}\right)^2\right]^2 + z^2}}$$

with the resonance peak $V_{max} = \frac{1}{z}$. The comparison indicates that the effect of the pendulous mass on the system is simply founded on a displacement of the natural frequency from ω_e toward ω_e'.

For the practical important case of $q = p$, called the resonant tuning of the pendulous mass, $\omega_e' = 0$ and $V = 0$. In other words, the mass of the system no longer executes deflections at all within the entire rotative speed range. The deflections of the pendulous mass in this instance (cf. appendix 3) are

$$\boxed{b = \frac{P}{m_o\ \omega^2}} \tag{6}$$

or in other words, the mass force of the pendulous mass $m_o b\ \omega^2$ holds the excitation P in equilibrium and cancels it to a certain extent.

This case is reminiscent of the known phenomenon of the double pendulum and of the elastically joined dynamic vibration absorber tuned to resonance with which the deflections of the system can be reduced to zero for a specified frequency. But, while this additional mass is in resonance at one certain rotative speed only, the above tuned pendulous mass is in resonance at every speed of rotation, because its natural frequency is proportional to the rotative speed. The pendulous mass therefore adapts itself to the particular type of excitation of a power plant, at which the frequency of excitation itself is proportional to the rotative speed.

For other cases of tuning, figure 6 shows the displacement of the natural frequency according to equation (4), and figure 7 several selected resonance curves. The relative deflections b/a can be read off from figure 12. The resonance curves were plotted for $\frac{m_o}{m} = \frac{1}{5}$ and $z = \frac{1}{15}$, that is, where the pendulous mass is the fifth part of the mass of the system and the damping force amounts to 6.5 percent of the spring force, giving a resonance amplification of $V = 15$. These figures are approximately correct for the conditions in a radial engine. For tuning $\frac{p}{q} < 1$ the natural frequency diminishes, for $\frac{p}{q} > 1$ it increases. In the range $p/q = 1$ to 1.1 or generally for any $\frac{m_o}{m}$ within range of $p/q = 1$ to $p/q = \sqrt{1 + \frac{m_o}{m}}$ no resonance occurs (see the curves for $p/q = 1$, 1.05, and 1.095 in fig. 7), although the system executes small deflections inferior to those of the static deflection. Mathematically, this case can be represented by visualizing the normal resonance curve continued into the zone for $(\omega/\omega_e)^2 < 0$ in which it progressively drops from $V = 1$ for $\omega/\omega_e = 0$ to $V = 0$ for $(\omega/\omega_e)^2 = -\infty$. This zone happens here to lie in the normal rotative speed range.

At only one harmonic excitation of a specific order acting on the mass does the case of resonant tuning in a one-mass system at which the deflections are zero, become of importance. (On the multimass system still other tunings are of interest, as will be shown elsewhere.) But, if several harmonic forces of different orders are active simultaneously, or in other words, if some periodic force acts exciting on the system, two different tunings of the pendulous mass are concurrently in existence, since the pendulous mass can be tuned to only one order of resonance. In two-stroke cycle engines, for instance, all whole numbered, in four-stroke cycle engines all half numbered orders of excitations are possible in addition. Concerning the behavior of the pendulous mass in such an event and the resultant vibrations of the system, a glance at figure 4 shows that the system for a given rotative speed does not differ in the least from a common two-mass system, and the deflections caused by the individual harmonic excitations are undisturbedly superimposed. Since the individual orders correspond to different pendulum mass tuning, every order has a different natural frequency: the system has a different natural vibration frequency for every order of tuning. The procedure consists in defining the resonance for the particular order and the subsequent superposition of the individual curves.

A resonance curve of a system involving several critical resonance points will undergo a displacement of such points because of the pendulous mass, where by appropriate tuning one can be completely eliminated. The amount of displacement is largely dependent upon the value m_0/m, the displacement being so much greater as the pendulous mass is greater in comparison with the mass of the system. Figure 8 is a schematic view of the resonance curve of a one-mass system embodying three critical speeds of rotation, that is, with and without a pendulous mass tuned to the second order, which approximately corresponds to the conditions arising in a single-cylinder two-stroke cycle compression-ignition engine; m_0/m is again assumed equal to 1/5. The critical of the second order disappears, that of the third shifts slightly upward, that of the first downward.

IV. ARBITRARY VIBRATION SYSTEM WITH ARBITRARY PENDULUM MASSES

Visualize a pendulum mass of the type of figure 2 fitted at some point O, consisting of any chosen number of masses and forming part of some torsional vibration system, as in figure 9. This pendulum mass, at point O, is to participate on the rotary motion of the system of angular speed Ω.

In conformity with the foregoing arguments, the pendulum mass can be replaced, according to figure 4, by a mass with the stiffness $c_0\Omega^2$ flexibly coupled to the system so as to form the substitute pendulum mass of figure 10, consisting of two masses m_1 and m_2, the former rigidly connected with the system, the latter coupled across the spring $c_0\Omega^2$. Subsequently, it will be shown that any kind of pendulum mass (till now, only a mathematical pendulum has been considered) can be represented by this two-mass substitute.

The natural frequency of the mass m_2 relative to the hypothetical static mass m_1 is

$$\omega_{e0}^2 = \frac{c_0\,\Omega^2}{m_2}$$

hence the resonance:

$$\boxed{q^2 = \frac{c_0}{m_2}} \tag{7}$$

Then assume that mass m_1 executes a positive vibratory motion $a_1 \sin \omega t$ of amplitude a_1 and frequency ω. Which is the motion and which the reaction of the pendulum mass on m_1 and hence on the system?

Since no damping acts at m_2 it swings in the same phase as m_1 otherwise no force equilibrium would be possible. The same was the case illustrated in figure 5. The simple force diagram for the present case is shown in figure 11.

The equilibrium condition for m_2 gives

$$c_0 \Omega^2 (a_2 - a_1) = m_2 a_2 \omega^2$$

$$\frac{a_2 - a_1}{a_2} = \frac{\omega^2/\Omega^2}{c_0/m_2} = \left(\frac{p}{q}\right)^2$$

$$\boxed{\frac{a_2}{a_1} = \frac{1}{1 - (p/q)^2} = \frac{\varphi_2}{\varphi_1}} \tag{8}$$

and for the relative deflection $b = a_2 - a_1$

$$\frac{\psi}{\varphi_1} = \boxed{\frac{b}{a_1} = \frac{1}{(q/p)^2 - 1}} \tag{9}$$

For the resonance tuning $q = p$, there is obtained

$$a_2 = b = \infty \quad \text{if} \quad a_1 \neq 0$$

The reaction on the system then consists of a force $W \sin \omega t$ which follows from the diagram at

$$W = m_1 a_1 \omega^2 + c_0 \Omega^2 (a_2 - a_1)$$

$$= m_1 a_1 \omega^2 + m_2 a_2 \omega^2$$

$$\boxed{W = \left(m_1 + \frac{m_2}{1 - (p/q)^2}\right) a_1 \omega^2} = M a_1 \omega^2 \tag{10}$$

This force can be regarded as mass force caused by a mass M at point O of the amount of

$$\boxed{M = m_1 + \frac{m_2}{1 - (p/q)^2}} = M_1 + M_2 \tag{11}$$

With this substitute mass affixed to the system, the total effect of the pendulum mass on the system is defined for a given mode of excitation. Here also the pendulum mass merely affects the natural vibration, that is, as regards frequency and mode of vibration. The substitute mass consists of a constant portion $M_1 = m_1$ and a portion $M_2 = \frac{m_2}{1 - (p/q)^2}$ varying with the tuning of the pendulum mass.

It is

$$\frac{M_2}{m_2} = \frac{a_2}{a_1} = \frac{b}{a_1} + 1$$

Figure 12 shows $\frac{M_2}{m_2}$ plotted against p/q, concurrently with $\frac{a_2}{a_1}$ and $\frac{b}{a_1}$ in function of p/q.

For tuning near $p/q = 1$ the substitute mass M_2 and hence the reaction of the pendulum mass on the system is very sensitive to small variations in tuning.* Any strong effect on the system by small pendulum mass is on the whole confined to the range of $p/q = 0.9$ to $p/q = 1.1$, that is, the practical range mostly.

The mass M_2 can assume any value between $+\infty$ and $-\infty$ with exception of the small range of $M_2 = 0$ to $M_2 = m_2$.

For $q = p$, $M = \infty$. That is, this tuning acts for the related order exactly as an infinitely large flywheel (dynamic flywheel) attached at point O. The deflections at the particular point thus become zero, and a node results. But the deflections of the rest of the system are not removed by it. The reaction of the pendulum mass in this case consists of the force

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*The effect of certain "corrections" to resonant tuning, established by experiment with which Salomon obtained for certain cases "the best results" by trial, is probably also traceable to it.

$$W = m_2 \, a_2' \cdot \omega^2$$

or

$$\boxed{W = m_2 \, b' \, \omega^2} \tag{12}$$

A deflection b' will result such that the reaction exactly reaches the value necessary for maintaining the vibration node. From it the relative deflection can be calculated. In a one-mass system the reaction is equal to the excitation $W = P$ acting on the mass.

For $p/q > 1$, M_2 becomes negative! The reaction then corresponds in its phase to a spring force acting contrary to the mass force, that is, to a substitute spring in a certain degree. But the amount of the reaction grows like a mass force with the square of the frequency. The pendulum mass here swings opposite to the mass of the system. Through this tuning, which results in a negative substitute mass, the natural frequency of the system can be raised, a possibility of extreme importance in practice. While the lowering of the natural frequency of a system is usually accomplished very easily by employing additional masses, the opposite case, the raising of the natural frequency is often beset with difficulties because a reduction in the available masses or a stiffening of the shaft system is not always possible. Here the pendulum mass with this tuning equivalent to a substitute spring constitutes a very practical means. The effect is, of course, for the present, restricted to a given mode.

For the mathematical prediction of the natural frequencies, by Gümbel's method, for instance, the negative substitute mass enters the calculation as negative quantity, which in nowise affects the calculation process and affords a clear representation.

The relations are predicated on a frictionless motion of the pendulum mass. By a small damping of the pendulum mass motion, the substitute mass in resonance will probably reach a very high value, but not as great as ∞. Assuming the damping force of the pendulum mass at $k_0 \, \Omega^2 \, b$, that is, proportional to the deflection and to the bearing load due to centrifugal force, the substitute mass M_2 becomes, (as shown in appendix 4):

$$M_2 = \frac{m_2}{\sqrt{\left(1 - \frac{p^2}{q^2}\right)^2 + z_0^2}}$$

if $z_o = \frac{k_o}{c_o}$ is the proportionate damping. The change in sign is effected through the phase change. At resonant tuning, $M_2 = \frac{m_2}{z_o}$. With a 0.1 percent damping, which may be assumed for a rolling resistance in order of magnitude, the substitute mass would in this instance become 1000 times as great as the actual mass. In its reaction on the system, this mass should differ very little from an ∞ size mass (fig. 13). Since only rolling friction is involved in practical pendulum masses and the roller tracks themselves are hardened for this purpose, the damping is very small. It is therefore disregarded and the subsequent study continued on the basis of frictionless motion. Since the choice of tuning makes it possible to accord almost any positive or negative value to the substitute mass, the natural frequencies for a given mode can be manipulated within very wide limits. The natural frequency of a multimass system approximating a six-cylinder engine with flywheel for different values of substitute mass M is shown in figure 13. The fundamental frequency without substitute mass, put at $\omega_e = 1$, can be varied through M within the limits of about $\omega_e = 0.23$ and 2.06, that is, across the wide range of about 1:9. The natural frequency approaches the limits asymptotically and on exceeding them, changes to the natural frequencies of a different degree, that is, with more or less nodes. In the transitional points a node is formed in the place of the pendulum mass. The higher natural frequencies are, of course, simultaneously affected by the mass M.

Summing up, it is found that with the pendulum mass for a given mode it is possible to

1. Apply, to a certain degree, an infinitely great mass at a point which reduces the deflections to zero,

2. Reduce an existing great mass to zero,

3. Provide any large positive or negative masses.

In this way, critical resonance zones can be considerably displaced if it is deemed desirable and shifted outside of the operating range under certain certain conditions. But the change in natural frequency likewise changes the mode of the natural frequency, hence also affects the excitation and damping of the system. So, for instance, it is possible to manipulate the vibration mode of a power plant in such a way that certain vibrations in cylinders cancel each other and no vibration from this source develops. By changing the mode of vibration of the power plant of a ship the proportionate deflections of the propeller can be increased and so the great damping of the propeller can be utilized to better advantage. Or an elastic coupling could be utilized more for damping by increasing its proportionate relative deflections. By fitting an ∞ size dynamic flywheel to a multicylinder engine, the transfer of vibrations from the engine on the driven side (such as generator, propeller) can be eliminated for a given mode. This applies in particular also to the case where a node already exists without flywheel and the pendulum mass cannot influence the natural frequency at all. It is a known fact that on a large flywheel located in the node of a vibration, the resonance deflections on either side of the flywheel differ considerably from the natural vibration mode.

In the event that several harmonic excitations of different modes act on the system simultaneously, or a periodic force is involved, every mode must be analyzed by itself. The tuning and the substitute mass of the pendulum mass is defined for each mode and with this the natural vibration of the system computed.

If the system is provided with several pendulum masses which may be disposed at one or various points and with different tuning, each pendulum mass for a given mode can be replaced by its substitute mass. A mutual disturbance of the pendulum masses in their behavior is not possible. With these different substitute masses the natural vibrations are computed for one mode and then repeated for others. Ordinarily the number of dangerous modes is few.

The application of several pendulum masses makes it possible to manipulate several resonances as to position and intensity and so to eliminate under certain conditions all critical zones within the operating range. No general rules can be given; each vibration system must be treated according to its own particular nature.

V. FUNDAMENTAL FORMS OF ROTATING PENDULUM MASSES

So far, a point-pendulum mass, a mathematical pendulum, had been assumed. In the following various practical design versions of pendulum masses are analyzed with regard to their

1. Tuning

2. Reaction to the system or substitute system, and

3. Relative deflections

The knowledge of the relative deflections is necessary for the choice of size of the pendulum mass, for the disposition of stops and for an estimate of the extent to which the theoretical study retains its validity for the actual deflections.

Subsequent to having established their tuning q all pendulum masses are reduced to the substitute system of figure 10, the equations of which are used for the study and which read as follows:

the reaction $\boxed{W = M a_1 \omega^2}$ whereby $\equiv (10)$

the substitute mass $\boxed{M = m_1 + \frac{m_2}{1 - (p/q)^2}}$ $\equiv (11)$

the relative deflections $\boxed{\frac{\psi}{\varphi_1} = \frac{b}{a_1} = \frac{1}{(q/p)^2 - 1}}$ $\equiv (9)$

for the resonant tuning $q = p$ there is obtained

$$a_1 = 0 \qquad M = \infty$$

and the reaction in this case

$$\boxed{W = m_2 b' \omega^2} \equiv (12)$$

1. Mathematical Pendulum

Supposing the pendulum illustrated in figures 2 and 3 has the mass m_0 and is attached to a massless disk ($\Theta = m = 0$).

From the equality of the restoring force and the mass force the tuning follows at

$$m_0 (r + s)\Omega^2 (\epsilon - \psi) = m_0 \psi (r + s) \omega_{eo}^2$$

with

$$\epsilon - \psi = \psi \frac{r}{s}$$

$$\left(\frac{\omega_{eo}}{\Omega}\right)^2 = \frac{r}{s}$$

hence

$$\boxed{q^2 = \frac{r}{s}} \qquad (1)$$

The substitute mass at lever arm $R = r + s$ follows according to equation (11), whereby $m_1 = 0$ and $m_2 = m_0$, at

$$\boxed{M = \frac{m_0}{1 - (p/q)^2}} \qquad (13)$$

The angular deflection ϵ of the pendulum mass about its point of suspension O' at a deflection φ_1 of the system about O follows from

$$\frac{\psi}{\varphi_1} = \frac{1}{(q/p)^2 - 1}$$

with $\epsilon = \psi \dfrac{r + s}{s} = \psi(q^2 + 1)$ becomes

$$\boxed{\frac{\epsilon}{\varphi_1} = \frac{q^2 + 1}{(q/p)^2 - 1}} \qquad (14)$$

For the resonant tuning $q = p$, φ_1 becomes 0 and ϵ for the present, indeterminate. Under a force W acting on the system (disk) in O at lever arm R, ϵ assumes a value ϵ', secured from equation (12)

$$W = m_0 \psi' R \omega^2$$

$$\epsilon' = \frac{W}{m_0 R \omega^2}(q^2 + 1)$$

or

$$\boxed{\epsilon' = \frac{W}{m_0\, r\, \Omega^2}} \qquad (15)$$

The formula indicates that ω' is so much smaller by equal force W as the speed of rotation is greater.

2. Design Version Wright (fig. 14)

This pendulous counterweight used by the Wright company in its radial engine is roller supported at two pivot points by two pins. If the disk is held stationary the mass can execute only a pure displacement motion, each point describing a circle of radius $s = d_1 - d_2$, where d_1 is the hole diameter, d_2 the pin diameter. The pendulum mass can, in this case, be replaced by a mass point in S, which is pendulously linked in O' and dynamically corresponds to the previously investigated mathematical pendulum. Its tuning is therefore also

$$\boxed{q^2 = \frac{r}{s}} \qquad \equiv (1)$$

If the disk itself performs a rotary motion, the pendulum mass executes likewise a rotary motion of the same amount when point S is held stationary. Its moment of inertia about S must therefore be added to the system. Hence with m_0 as the mass and $m_0\, i^2$ as the inertia moment of the pendulum mass about its point S (i – arm of inertia), the substitute mass is

$$M = M_1 + M_2$$

$$\boxed{M = m_0\left(\frac{i}{R}\right)^2 + \frac{m_0}{1 - (p/q)^2}} \qquad (16)$$

referred to the lever arm $R = r + s$. The mass of the pin can be disregarded.

The values for the relative deflections are the same as for the mathematical pendulum

$$\boxed{\frac{\epsilon}{\varphi_1} = \frac{q^2 + 1}{(q/p)^2 - 1}} \qquad \equiv (14)$$

and the resonant tuning $q = p$

$$\boxed{\epsilon' = \frac{W}{m_0 \, r \, \Omega^2}} \equiv (15)$$

as before.

The wright design has the great advantage of involving rolling friction only. The reactive forces in the bearings always pass perpendicularly through the contact point of the bearing surfaces so that no sliding can occur even at large vibratory deflections ϵ. A further advantage is that the tuning can be made as high as considered desirable since s can be made arbitrarily small (in contrast to the material pendulum).

3. Material Pendulum

Let the pendulum, illustrated in figure 15, have the mass m_0 and a moment of inertia $m_0 \, i^2$ about its point S. A body, by its linear motion, can be represented dynamically by two mass points having the same point S, the same total mass, and the same moment of inertia. The pendulum is accordingly replaced by two mass points m_1 and m_2 of which m_1 is located in the pivot point O'.

When s' is the distance of mass m_2 from O', the conditions

$$m_1 \, s = m_2 \, (s' - s)$$

$$m_1 + m_2 = m_0$$

$$m_1 s^2 + m_2 (s' - s)^2 = m_0 \, i^2$$

afford

$$s' = s \left[1 + (i/s)^2 \right] \qquad \boxed{m_1 = \frac{m_0}{1 + (s/i)^2}} \qquad \boxed{m_2 = \frac{m_0}{1 + (i/s)^2}} \qquad (17)$$

s' corresponds, moreover to the reduced pendulum length of a material pendulum.

Herewith the pendulum mass is reduced to a mathematical pendulum m_2 the length of which is s' and to an additional mass m_1 rigidly connected with the system. The tuning is

$$\boxed{q^2 = \frac{r}{s'} = \frac{r}{s\left(1 + \frac{i^2}{s^2}\right)}} \tag{18}$$

and the substitute mass on lever arm $R = r + s' = r + s + \frac{i^2}{s}$

$$\boxed{M = m_1 \left(\frac{r}{R}\right)^2 + \frac{m_2}{1 - (p/q)^2}} \tag{19}$$

m_1 and m_2 to be computed according to (17). It is to be noted that the reduction involves lever arm $R = r + s'$ rather than $R = r + s$. The relative deflections ϵ are, as on the mathematical pendulum:

$$\boxed{\frac{\epsilon}{\varphi_1} = \frac{q^2 + 1}{(q/p)^2 - 1}} \equiv (14)$$

for $q = p$ $\qquad \epsilon' = \frac{W}{m_2 R \omega^2} (q^2 + 1)$

or

$$\boxed{\epsilon' = \frac{W}{m_2 r \Omega^2}} \tag{20}$$

For the practical application the material pendulum has the disadvantage of attendant sliding motion in the bearing and the impossibility of tuning the pendulum to high frequencies since s' cannot be constructed below a given value. It always is

$$\boxed{s' \geqq 2 i}$$

4. Roller According to Salomon

a) Outside roller. This pendulum mass (fig. 16) is formed by a ring which unrolls on a fixed pin, distance r from the axis of rotation. The outside roller is therefore not pivotally mounted in a point O' but roller supported on a pin. The pin has a radius ρ_1, the inside roller track of the ring the radius ρ_2, a mass m_0 and an inertia moment $m_0 i^2$ (fig. 17). The pendulum mass is again replaced by two mass points, of which m_1 is placed in the point X, the contact point between pin and ring in equilibrium position of the ring; X forms the momentary center of the motion for small deflections, but the center of the path of the centroid is O'. It is

$$\boxed{m_1 = \frac{m_0}{1 + (\rho_2/i)^2} \quad\Big|\quad m_2 = \frac{m_0}{1 + (i/\rho_2)^2}} \qquad (21)$$

and the distance m_2 from S

$$u = \frac{i^2}{\rho_2}$$

The mass m_1 must here also be added to the mass of the system as it joins in its motion. But now mass m_2 can no longer be regarded as self-contained mathematical pendulum subjected to the centrifugal force since m_1 does not form the point of suspension for m_2. At deflections of the pendulum mass, the point of support shifts toward contact point Y so that the centrifugal force of mass m_1 then contributes also to the restoring forces of mass m_2.

When the system executes no vibratory motion ($\varphi_1 = 0$) the pendulum mass is subjected to the effect of three forces: the centrifugal force in the center of mass S, the mass force of the vibratory motion in m_2, and the bearing reaction due to Y, which must be in equilibrium with each other. The bearing reaction must accordingly pass through the point of intersection Z of mass force and centrifugal force.

The equilibrium in point Z demands

$$F \gamma = m_2 b \omega_{eo}^2$$

which corresponds to the equation: restoring force = mass force.

The quantities γ and b can be secured from geometrical relations, after which the insertion of all the values (cf. appendix 5) affords

$$\omega_{eo}^2 = \frac{r}{s[1 + (i/\rho_2)^2]} \Omega^2$$

and consequently

$$\boxed{q^2 = \frac{r}{s[1 + (i/\rho_2)^2]}} \tag{22}$$

The substitute mass at lever arm $R = r + s + u = r + s + \frac{i^2}{\rho_2}$ is

$$\boxed{M = m_1\left(\frac{r - \rho_1}{R}\right)^2 + \frac{m_2}{1 - (p/q)^2}} \tag{23}$$

m_1 and m_2 being obtained from equation (21).

The relative deflections for m_2 are again

$$\frac{\psi}{\varpi_1} = \frac{1}{(q/p)^2 - 1}$$

The relation between ψ and ϵ affords (cf. appendix 5)

$$\epsilon = \frac{R}{r} q^2 \psi$$

with

$$R = r + s + \frac{i^2}{\rho_2}$$

hence

$$\boxed{\frac{\epsilon}{\varpi_1} = \frac{R/r\ q^2}{(q/p)^2 - 1}} \tag{24}$$

at resonance $q = p$

$$\psi' = \frac{W}{m_2 R \omega^2}$$

therefore

$$\boxed{\epsilon' = \frac{W}{m_2 r \Omega^2}} \equiv (20)$$

According to figure 17, the bearing reaction in Y is not perpendicular to the contact area but inclined at an angle η to the normal. At large deflections, therefore, a sliding can occur which must be avoided here under all circumstances. At the high bearing pressures and rapid oscillations, such sliding at this point cannot be controlled by lubricating technique.

Angle η, which must always be smaller than the angle of friction and which is composed of the angle η_2 plotted in figure 17 and an angle η_1 conditioned by the vibratory motion of m_1, follows from geometrical relations (cf. appendix 6) at

$$\boxed{\eta = \frac{p^2 (r - \rho_1) \varphi_1 - r\epsilon}{(r + s)(1 + \rho_2^2/i^2)}} \tag{25}$$

wherein either φ_1 or ϵ could be posted according to equation (24).

At resonance $q = p$ for $\varphi_1 = 0$

$$\boxed{\frac{\eta'}{\epsilon'} = \frac{r}{r + s} \frac{1}{1 + (\rho_2/i)^2}} \tag{26}$$

b) Inside roller.- For the inside roller of figure 18 where one roller can swing within a hole, the geometrical proportions are represented in figure 19. The notation is the same as in figure 17. The derivation is the same as for the outside roller. However, the relations for the inside roller are obtained from those for the outside roller by simply posting ρ_1 and ρ_2 negative in conformity with the geometrical inversion.

Tuning and relative deflections remain unaltered.

The substitute mass at lever arm $R = r + s - \frac{i^2}{\rho_2}$ is

$$\boxed{M = m_2 \left(\frac{r + \rho_1}{R}\right)^2 + \frac{m_2}{1 - (p/q)^2}} \tag{27}$$

and η

$$\boxed{\eta = \frac{p^2 (r + \rho_1) \varphi_1 - r\epsilon}{(r + s)(1 + \rho_2^2/i^2)}} \tag{28}$$

η'/ϵ' remains unchanged.

The cited equations (22) and (20) for the tuning and the deflection ϵ' at resonance are in agreement with those obtained by Salomon by a different method.

5. Pendulum Mass with Two Degrees of Freedom

All the pendulum masses discussed so far had, in comparison with the system, only one degree of freedom, if the possibility of lifting from the support, prevented by the centrifugal forces, is discounted. Figure 20 shows a pendulum mass with two degrees of freedom. It comprises a pendulum mass of the Wright type with one of the supports omitted, or of Salomon's outside roller with the support pin itself mounted on rollers.

Since the pendulum mass has two degrees of freedom, it is to be expected that it will also have two different natural frequencies and two different tunings. In consequence it should be possible with such a pendulum mass to manipulate two different modes simultaneously.

Figures 20 and 21 show the mass at the instant of maximum deflection. The mass may be visualized as having been put in this position by giving it, first, a displacement (similar to Wright), at which point S makes an angular deflection ϵ about O' (O', O" and S then still lie on a straight line); then the mass is slightly rolled on the stationary pin, the S point making an angular deflection δ about O" (δ does not correspond to the angle of rotation). The position of the mass is defined by ϵ and δ. The conversion in the position ϵ, δ, on the other hand, however, corresponds to a temporarily indeterminate point X on the axis OO', with distance of, say, xs = v. Then the pendulum mass is again replaced by two mass points m_1 and m_2, the first, m_1 being located in X. The pin ρ_2 is assumed to be massless.

Now there are three forces acting on the pendulum mass, the centrifugal force in S, the mass force in m_2, and the bearing reaction which passes through the contact points YY' of the pin, since the pin is massless. Equilibrium is possible only when the bearing reaction passes through the intersection Z of the two other forces. This condition leads, by reason of the geometrical relations (appendix 7), to the unshortened equation

$$\frac{(\epsilon + \delta/2)\ \delta/2}{(\epsilon + \delta/2)\ \delta/2} = \left(\frac{i}{\rho_1}\right)^2 \frac{r}{r + s}$$

which can be satisfied for any given values of i, ρ, r, and s only when either

I	$\delta = 0$	then I	$v = \infty, \quad u = 0$
or II	$\delta = -2\epsilon$	or II	$v = 0, \quad u = \infty$

This means that the mass executes either a displacement motion ($v = \infty$) or a rotary motion about S. These are the two possible states of vibration of the mass. They are illustrated in figure 22.

In the first instance the mass moves exactly like Wright's pendulum mass and all of Wright's formulas for q, M, ϵ, and so forth, hold true, including, in particular,

$$\boxed{q_I^2 = \frac{r}{s}} \qquad \equiv (1)$$

In the second case the tuning follows from the moment equilibrium about S, restoring force = mass force,

$$F \rho_1 \epsilon = m_0 i^2 \alpha \omega^2$$

or

$$m_0 (r + s) \Omega^2 \rho \epsilon = m_0 i^2 \epsilon \frac{s}{\rho_1} \omega^2$$

at

$$\boxed{q_{II}^2 = \left(1 + \frac{r}{s}\right)\left(\frac{\rho_1}{i}\right)^2} \qquad (29)$$

For the substitute mass at distance $R = r + s$ there is obtained

$$m_1 = m_0 \text{ in } S$$

and $m_2 = 0$ in ∞; whereas the moment of inertia $m_0 i^2$ is therefore

$$\boxed{M_{II} = m_0 + \frac{m_0 (i/R)^2}{1 - (p/q_{II})^2}} \qquad (30)$$

with

$$\psi = \epsilon \frac{s}{\rho_1}$$

equation (9) gives the relative deflections at

$$\boxed{\frac{\epsilon_{II}}{\varphi_1} = \frac{\rho_1/s}{(q_{II}/p)^2 - 1}} \quad \text{and} \quad \delta = -2\epsilon \qquad (31)$$

ϵ_{II}' for the resonant tuning can here also be derived from equation (12), but it is simpler obtained from figure 22, where the reaction consists of the moment

$$m_o (r + s) \Omega^2 \rho_1 \epsilon' = W (r + s)$$

whence

$$\boxed{\epsilon' = \frac{W}{m_o \rho_1 \Omega^2}} \tag{32}$$

From the geometrical pattern it is seen that

$$\boxed{\eta_{II} = - \epsilon}$$

For the case of

$$\left(1 + \frac{s}{r}\right) \left(\frac{\rho_1}{i}\right)^2 = 1$$

there is obtained

$$q_I = q_{II}$$

The pendulum mass with two degrees of freedom, which like the Wright version and Salomon's roller possesses only rolling friction can be tuned simultaneously to two different modes or orders.

Figure 23 is a design sketch for the simultaneous disposition of four such pendulum masses. This device flanged to the shaft can be tuned to four different modes simultaneously if the pendulum masses facing each other are designed in pairs as in the sketch; if the opposite masses are designed dissimilar, theoretically, a device with eight different modes of tuning simultaneously is possible. The springs serve to keep the masses in their position when at rest. Because of their smallness their restoring force is secondary. The term damper is purposely avoided for such a device which in a way operates as frequency transformer.

Eventually, existing counterweights could likewise be designed with advantage as pendulum masses.

6. Compilation of Formulas and General Remarks.

The table in the appendix contains all the data of interest for the aforementioned types of pendulum masses. Other possible types are not discussed here. They can be analyzed in the same manner by substituting two mass points, one of which is placed in the fictitious point of rotation of the vibratory motion. If the tuning at a pendulum mass is to be ascertained only, this can usually be obtained in simpler manner. In such a case only the center of gravity of the body is considered, and the actual centrifugal force is allowed to act on a fictitious mass having the same kinetic energy as the pendulum body. On rollers, for instance, this mass is $m_0\left(1 + \frac{i^2}{l^2}\right)$ where l is the distance of the center of mass from the point of support.

One instance of pendulum mass of the Salomon type concerns the case where the center of mass does not coincide with the center of its path of rolling. In this instance the tuning is

$$\boxed{q^2 = \frac{r + s}{l\left(1 + \frac{i^2}{l^2}\right)}\left[1 - \frac{l}{r + s} + \frac{\rho_1 \rho_2}{l(\rho_2 - \rho_1)}\right]} \qquad (33)$$

whereby $l = s + \rho_1$.

For $l = \rho_2$ or $\rho_2 - \rho_1 = s$ the formula changes to the equation (22) for the concentric roller. With $\rho_1 = 0$ or $\rho_2 = s$ the formula (18) for the material pendulum is obtained.

The design of a pendulum mass is above all predicated on the knowledge of the relative deflections ϵ to be expected and the angle of inclination η. The magnitude of the pendulum mass is governed by it. The greater the permitted deflections the smaller the pendulum mass itself can be chosen. For large deflections ϵ the mathematical assumptions are no longer fulfilled so that the pendulum mass has, under certain conditions, a tuning other than intended. Salomon quotes $\epsilon = 15^\circ$ as upper limit value. Experience must decide it. For a nine-cylinder radial engine this deflection on a pendulum mass of the Wright type would only amount to about 6° with $m_0/m = 1/5$, and that is for the worst case that the main critical lies at full throttle speed of rotation.

The angle of inclination η that arises on certain pendulum masses must always be smaller than the angle of friction in order to positively avoid sliding. In the extreme case, 0.1 equivalent to an angle of 6° might be permitted.

VI. SUMMARY

The rotating pendulum mass subjected to centrifugal force corresponds in its effect to a mass elastically combined with the system, whereby the spring stiffness is dependent on the rotative speed.

The resonant vibrations of a one-mass system excited by a periodic force of given mode can be completely removed by a pendulum mass tuned exactly to resonance. If the pendulum mass is not tuned to resonance the natural frequency of the system of the particular mode can be manipulated and shifted within any limits upward or downward. If several modes of the exciting force exist simultaneously the system attains through the pendulum mass a different natural frequency for each mode. The individual resonance curves of the different modes are superimposed undisturbedly.

For the general case of an arbitrary multimass system, the reaction of any arbitrarily tuned pendulum mass disposed at some point for a given mode of excitation can be best described by a sutstitute mass rigidly connected to the system at the particular point. By appropriate tuning of the pendulum mass this substitute mass can be accorded any arbitrarily great value of any prefix. In this manner the natural frequency of the system for the particular order can be shifted within wide limits.

An infinitely great substitute mass can be provided and a node imposed at the particular point of the oscillation. An already existing mass can be cancelled in its effect. In particular, it is possible to raise the natural frequency of the system by an equivalent substitute mass of negative sign which acts as a spring. This possibility of raising the natural frequency is of particular value because no other means to accomplish it are known. With the natural frequency, the vibration mode is also affected by the substitute mass and hence the excitation and damping of the system.

In the event that several excitations of different modes exist, the substitute mass for each mode gets a different value; hence the system a different natural frequency for each mode. The individual vibrations are undisturbedly superimposed.

With multi-pendulum masses each mass becomes fully effective without possible mutual interference. Each pendulum mass is replaceable by its substitute mass.

NOTATION

Mass $\left[\text{kg cm s}^2 \text{ or } \frac{\text{kg s}^2}{\text{cm}}\right]$

Θ, $m = \frac{\Theta}{R^2}$	moment of inertia of the mass of the system, and the mass reduced to lever arm R, respectively
m_0	mass of pendulum
$m_0 i^2$	moment of inertia of pendulum about its center of gravity
m_1, m_2	dynamically equivalent substitute mass points of the pendulum
$M = M_1 + M_2$	dynamically equivalent substitute mass of the pendulum at lever arm R

Spring constants – damping constants $\left[\text{kg cm and } \frac{\text{kg}}{\text{cm}}\right]$

$C, c = \frac{C}{R^2}$	spring constant of one-mass system for the torsional vibration and for the longitudinal vibration at lever arm R, respectively
k	damping constant of the damping force of the one-mass system proportional to the deflection
$c_0 \Omega^2$	spring constant of the substitute system of the pendulum mass

Deflections [cm]

$\varphi, a = \varphi R$	vibratory deflection of the mass of the system
$\varphi_0, a_0 = \varphi_0 R$	deflection of the point-pendulum mass
ψ, $b = \psi R$	relative deflection of point-pendulum mass with respect to the system

$\left.\begin{matrix}\varphi_1, a_1 = \varphi_1 R \\ \varphi_2, a_2 = \varphi_2 R\end{matrix}\right\}$	deflections of substitute masses m_1 and m_2
ϵ	relative deflection of center of mass of pendulum about its path center (suspension point on simple pendulum)
η	angle of inclination of support force with respect to the normal of the contact surface
$a_{st} = \frac{P}{c}$	static deflection of the system due to force P at lever arm R
$a', b', \psi', \epsilon', \eta'$	denote the corresponding values for the specific case of resonant tuning $q = p$

Natural frequencies [1/s]

Ω	rate of rotation, angular velocity of rotatory motion
ω	frequency of the exciting force P
ω_e	natural frequency of the system without pendulum mass
ω_{eo}	natural frequency of the pendulum mass relative to the hypothetical static system

Forces [kg]

$P \sin \omega t$	exciting force at lever arm R
β	phase of the exciting force relative to the vibration of the system
F	centrifugal force in the center of gravity of the pendulum mass
$W \sin \omega t$	reaction of the pendulum mass on the system at lever arm R

Design dimensions [cm]

$r = OO'$	distance of the axis of rotation from the point of suspension, or path center S of the pendulum mass

$s = O'S$ distance of the center of gravity, respectively, of the suspension point or S-path center of the pendulum mass (= length of pendulum of the mathematical pendulum)

ρ_1 radius of roller path of the disk combined with the system

ρ_2 radius of roller path of the pendulum mass

Ratios

$z = \frac{k}{c}$ damping of the one-mass system, ratio of damping force to spring force; the logarithmic decrement is then $\vartheta = \pi z$

$q = \frac{\omega_{eo}}{\Omega}$ tuning of pendulum mass, the ratio of its natural frequency to the rotative speed

$p = \frac{\omega}{\Omega}$ mode of excitation, ratio of exciter frequency to speed of rotation

$V = \frac{a}{a_{st}}$ magnification of the deflections of the system

Note: The bracketed dimensions refer, respectively, to the quantities of the torsional vibration and the corresponding quantities of the longitudinal vibration at lever arm, R.

Coordination of Formulas

	Mathemat. pendulum	Wright version	material pendulum	Outside roller Salomon	Inside roller Salomon	Roller with 2 degrees of freedom Case I	Case II
Fig.	2 and 3	14	15	16 and 17	18 and 19	20, 21 and 22	20, 21 and 22
q^2	$\frac{r}{s}$	$\frac{r}{s}$	$\frac{r}{s\left(1+\frac{i^2}{s^2}\right)}$	$\frac{r}{s\left(1+\frac{i^2}{\varrho_2^2}\right)}$	$\frac{r}{s\left(1+\frac{i^2}{\varrho_2^2}\right)}$	$\frac{r}{s}$	$\left(1+\frac{r}{s}\right)\left(\frac{\varrho_1}{i}\right)^2$
M	$\frac{m_0}{1-(p/q)^2}$	$m_0\left(\frac{i}{R}\right)^2+\frac{m_0}{1-(p/q)^2}$	$m_1\left(\frac{r}{R}\right)^2+\frac{m_2}{1-(p/q)^2}$	$m_1\left(\frac{r-\varrho_1}{R}\right)^2+\frac{m_2}{1-(p/q)^2}$	$m_1\left(\frac{r+\varrho_1}{R}\right)^2+\frac{m_2}{1-(p/q)^2}$	$m_0\left(\frac{i}{R}\right)^2+\frac{m_0}{1-(p/q)^2}$	$m_0+\frac{m_0\,(i/R)^2}{1-(p/q)^2}$
R	$r+s$	$r+s$	$r+s+\frac{i^2}{s}$	$r+s+\frac{i^2}{\varrho_2}$	$r+s-\frac{i^2}{\varrho_2}$	$r+s$	$r+s$
m_1	—	—	$\frac{m_0}{1+\left(\frac{s}{i}\right)^2}$	$\frac{m_0}{1+\left(\frac{\varrho_2}{i}\right)^2}$	$\frac{m_0}{1+\left(\frac{\varrho_2}{i}\right)^2}$	—	—
m_2	—	—	$\frac{m_0}{1+\left(\frac{i}{s}\right)^2}$	$\frac{m_0}{1+\left(\frac{i}{\varrho_2}\right)^2}$	$\frac{m_0}{1+\left(\frac{i}{\varrho_2}\right)^2}$	—	—
$\frac{\varepsilon}{\varphi_1}$	$\frac{q^2+1}{(q/p)^2-1}$	$\frac{q^2+1}{(q/p)^2-1}$	$\frac{q^2+1}{(q/p)^2-1}$	$\frac{\frac{R}{r}\,q^2}{(q/p)^2-1}$	$\frac{\frac{R}{r}\,q^2}{(q/p)^2-1}$	$\frac{q^2+1}{(q/p)^2-1}$	$\frac{\varrho_1/s}{(q/p)^2-1}$
ε'	$\frac{W}{m_0\,r\,\Omega^2}$	$\frac{W}{m_0\,r\,\Omega^2}$	$\frac{W}{m_2\,r\,\Omega^2}$	$\frac{W}{m_2\,r\,\Omega^2}$	$\frac{W}{m_2\,r\,\Omega^2}$	$\frac{W}{m_0\,r\,\Omega^2}$	$\frac{W}{m_0\,\varrho_1\,\Omega^2}$
η	—	0	—	$\frac{p^2\,(r-\varrho_1)\,\varphi_1-r\,\varepsilon}{(r+s)\left(1+\frac{\varrho_2^2}{i^2}\right)}$	$\frac{p^2\,(r+\varrho_1)\,\varphi_1-r\,\varepsilon}{(r+s)\left(1+\frac{\varrho_2^2}{i^2}\right)}$	0	ε
η'	—	0	—	$\frac{r}{r+s}\,\frac{\varepsilon'}{1+\left(\frac{\varrho_2}{i}\right)^2}$	$\frac{r}{r+s}\,\frac{\varepsilon'}{1+\left(\frac{\varrho_2}{i}\right)^2}$	0	ε'

APPENDIX 1

The force acting on M_0 in direction of the vibration (tangential), (fig. 3) is

$$F\ (\epsilon - \psi) = F\ \frac{r}{s}\psi\ .$$

At O' the force $F \cos (\epsilon - \psi) \approx F$ has a tangential component $F\ \epsilon$, which yields on m a force

$$F\ \epsilon\ \frac{r}{r + s} = F\ \frac{r}{s}\ \psi$$

Hence the two forces at m_0 and m act like a spring stretched between the two masses.

APPENDIX 2

$$\left| \begin{aligned} m_0\ a_0\ \omega^2 &= c_0\ \Omega^2\ b \\ ca &= P \cos \beta + c_0\ \Omega^2\ b + m\ a\ \omega^2 \\ ka &= P \sin \beta \end{aligned} \right|$$

From the first equation

$$\frac{b}{a_0} = \frac{b}{a + b} = \frac{m_0\ \omega^2}{c_0\ \Omega^2} = \frac{p^2}{q^2}$$

$$\frac{b}{a} = \frac{1}{(q/p)^2 - 1}\ \cdot\cdot\cdot$$

entered in the second equation

$$P \cos \beta = c\ a \left[1 - \frac{m\ \omega^2}{c} - \frac{c_0\ \Omega^2}{c\ [(q/p)^2 - 1]}\right]$$

with $c = m\ \omega_e^{\ 2}$ and $c_0\ \Omega^2 = m_0\ \omega_{eo}^2$

$$P \cos \beta = c\ a \left[1 - \frac{\omega^2}{\omega_e^2} - \frac{m_0}{m}\left(\frac{\omega_{e0}}{\omega_e}\right)^2 \frac{1}{(q/p)^2 - 1}\right]$$

or $$P \cos \beta = c\ a \left[1 - \left(\frac{\omega}{\omega_e}\right)^2 \left(1 + \frac{m_0}{m} \frac{1}{1 - (p/q)^2}\right)\right]$$

This equation and the third squared and added affords

$$P^2 = c^2\ a^2 [\text{——}]^2 + k^2\ a^2$$

whence

$$a = \frac{P/c}{\sqrt{[\text{——}]^2 + \left(\frac{k}{c}\right)^2}}$$

which with the notation gives the previously cited equation for V.

For the case without damping, $z = 0$, there is obtained

$$V = \frac{1}{1 - \left(\frac{\omega}{\omega_e}\right)^2 \left(1 + \frac{m_0}{m} \frac{1}{1 - (p/q)^2}\right)}$$

Taylor's equation reduced to the same notation and the one-mass system read on the contrary

$$V^* = \frac{1}{1 - \left(\frac{\omega}{\omega_e}\right)^2 \left(1 + \frac{m_0}{m} \frac{1}{[1 - (p/q)^2]\ (q^2 + 1)}\right)}$$

hence differs by the factor $(q^2 + 1)$, which is due to the erroneous premise.

APPENDIX 3

For $\frac{p}{q} = 1$ gives $\frac{b}{a} = \frac{0}{0}$

$\frac{b}{a_{st}} = \frac{b}{a} V$ gives with $p/q = 1$

$$\frac{b}{a_{st}} = \frac{1}{\left(\frac{\omega}{\omega_e}\right)^2 \frac{m_o}{m}}$$

or with $a_{st} = P/c$ and $\omega_e{}^2 = \frac{c}{m}$

$$b = \frac{P}{m_o\ \omega^2}$$

APPENDIX 4

The force diagram of figure 11 changes in this instance into that of figure 24, where the mass force in m_2 is vectorially divided along a_1 and b. The equilibrium gives

$$\left| \begin{array}{l} m_2\ a_1\ \omega^2 \cos\beta = c_o\ \Omega^2\ b - m_2\ b\ \omega^2 \\ m_2\ a_1\ \omega^2 \sin\beta = k_o\ \Omega^2\ b \end{array} \right|$$

squared and added

$$(m_2\ a_1\ \omega^2)^2 = (c_o\ \Omega^2\ b)^2\left[\left(1 - \frac{m_2\ \omega^2}{c_o\ \Omega^2}\right)^2 + \left(\frac{k_o\ \Omega^2}{c_o\ \Omega^2}\right)^2\right]$$

$$= (c_o\ \Omega^2\ b)^2\left[\left(1 - \frac{p^2}{q^2}\right)^2 + z_o{}^2\right] \quad \text{with} \quad z_o = \frac{k_o}{c_o}$$

The reaction of m_2 is $W = c_o\ \Omega^2 b$ or

$$W = \frac{m_2\ a_1\ \omega^2}{\sqrt{\left(1 - \frac{p^2}{q^2}\right)^2 + z_o{}^2}}$$

which, strictly speaking, may no longer be regarded as mass force since it does not fall in the direction of a_1. For small damping values z_o the directional departure is, however, small with exception of $q/p = 1$.

The damping force put at $k_o\ \Omega^2\ b$ is rather independent of the deflection by purely rolling resistance.

APPENDIX 5

Figure 17 affords

$$\gamma\ (\rho_2 + u) = (\epsilon - \zeta)\rho_2$$

and

$$\epsilon\ s = \zeta\ (r + s)$$

or

$$\epsilon - \zeta = \frac{r}{s}\ \zeta$$

with

$$u = \frac{i^2}{\rho_2}$$

becomes

$$\boxed{\gamma = \frac{r}{s}\ \frac{\zeta}{1 + \left(\frac{i}{\rho_2}\right)^2}}$$

further

$$b = (\rho_2 + u)\,\alpha$$

and

$$\alpha\ \rho_2 = (r + s)\ \zeta$$

hence

$$\boxed{b = (r + s)\left(1 + \frac{i^2}{\rho_2{}^2}\right)\ \zeta}$$

γ, b as well as F and m_2 posted in the equation

$$F\ \gamma = m_2\ b\ \omega_{eo}{}^2$$

and abbreviated gives equation (22).

It is

$$\psi R = \alpha\ (\rho_2 + u)$$

$$\alpha\ \rho_2 = \epsilon\ s$$

whence

$$\epsilon = \frac{\rho_2}{s}\ \frac{R}{\rho_2 + u}\ \psi$$

but

$$= \frac{R}{s\left(1 + \frac{i^2}{\rho_2{}^2}\right)}\psi = \frac{R}{r}\ q^2\ \psi$$

APPENDIX 6

Figure 17 affords

$$\eta_2 \rho_2 = \gamma u$$

$$\eta_2 = \frac{i^2}{\rho_2^2} \gamma$$

and appendix 5

$$\gamma = \frac{r}{s} \frac{\zeta}{1 + \frac{i^2}{\rho_2^2}}$$

with $$\zeta = \frac{s}{r + s} \epsilon$$

$$\gamma = \frac{r}{r + s} \frac{\epsilon}{1 + (i/\rho_2)^2}$$

hence $$\boxed{\eta_2 = \frac{r \epsilon}{(r + s)(1 + \rho_2^2/i^2)}}$$

The mass force $m_1(r - \rho_1) \varphi_1 \omega^2$ together with the support force F (small angles!) produces

$$\eta_1 = \frac{m_1 (r - \rho_1) \varphi_1 \omega^2}{m_0 (r + s) \Omega^2}$$

or $$\boxed{\eta_1 = \frac{(r - \rho_1) p^2 \varphi_1}{(r + s)(1 + \frac{\rho_2^2}{i^2})}}$$

$\eta = \eta_1 - \eta_2$ gives equation (25). (Sign not of interest!)

APPENDIX 7

The mass of the pendulum is subjected to a displacement of the S-point by $s(\epsilon + \delta/2)$ and a rotation through $\frac{\delta}{2} \frac{s}{\rho_1}$, which must be equal to the angle α. Whence

$$v = \frac{s\ (\epsilon + \delta/2)}{\alpha} = \rho_1 \frac{\epsilon + \delta/2}{\delta/2}$$

and consequently

$$\left| u = \frac{i^2}{v} = \frac{i^2}{\rho_1} \frac{\delta/2}{\epsilon + \delta/2} \right|$$

For angle γ between centrifugal force and bearing force, the triangle $OO'S$ – whereby $O'S$ // YY' –

$$\gamma = \epsilon + \delta/2 - \zeta$$

affords $\zeta(r + s) = s\ (\epsilon + \delta/2)$

hence $$\left| \gamma = \frac{r}{r + s}\ (\epsilon + \delta/2) \right|$$

In order that the point of intersection Z lie at distance u from S,

it must $u\ \gamma = \rho_1\ \delta/2$

$$\frac{i^2}{\rho_1} \frac{\delta/2}{\epsilon + \delta/2} \frac{r}{r + s} (\epsilon + \delta/2) = \rho_1\ \delta/2$$

that is $$\frac{(\epsilon + \delta/2)\delta/2}{(\epsilon + \delta/2)\delta/2} = \left(\frac{i}{\rho_1}\right)^2 \frac{r}{r + s}$$

Therefrom follow the further conclusions of the main text!

Translation by J Vanier,
National Advisory Committee
for Aeronautics.

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5. Stieglitz, Albert: Drehschwingungen in Reihenmotoren. Luftfahrtforschung, Bd. 4, Lfg. 5, July 24, 1929, pp. 133–158.

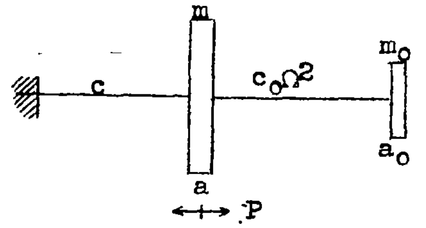

Figure 4.- Substitute system for figure 1.

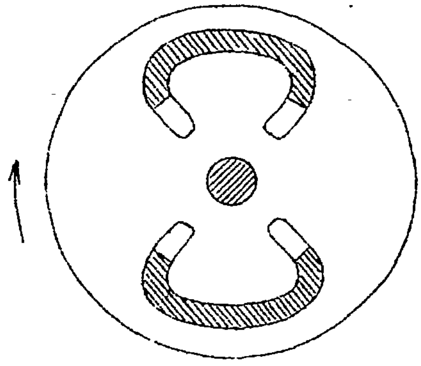

Figure 1.- Vibration damper (Kutzbach type).

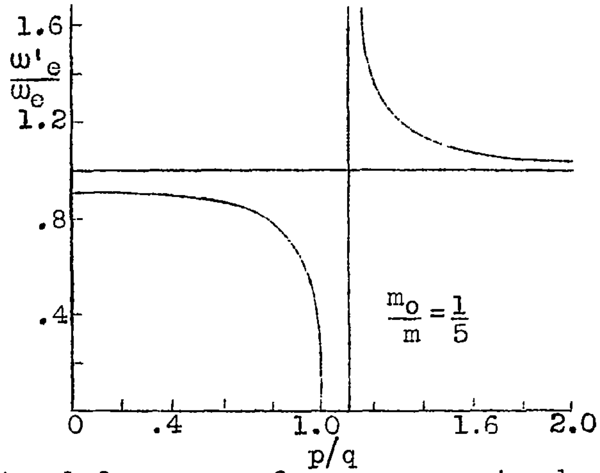

Figure 6.- Natural frequency of one-mass system by different tuning of pendulum mass.

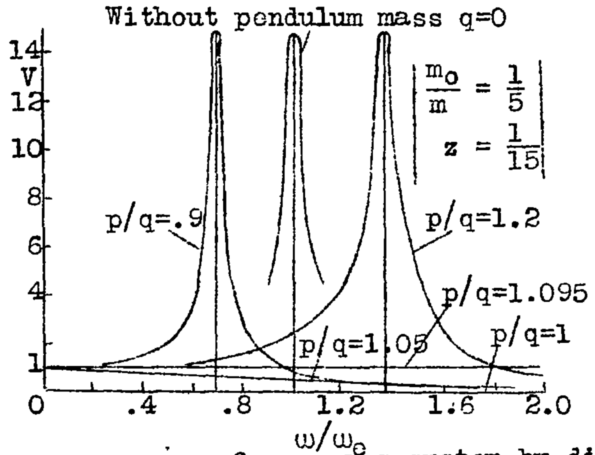

Figure 7.- Resonance curves of one-mass system by different tuning of pendulum mass.

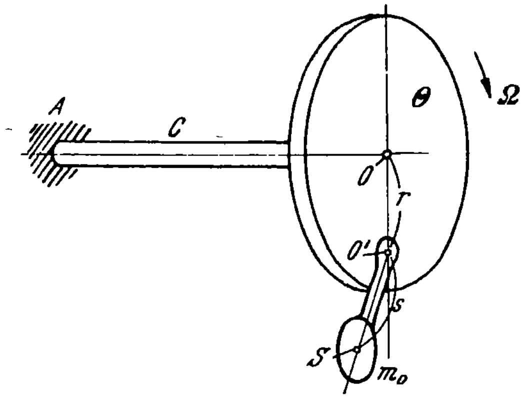

Figure 2.- Single-link vibration system with a pendulum mass.

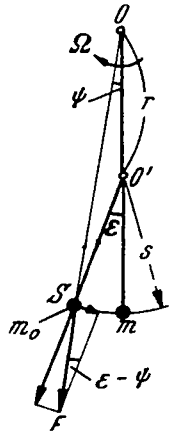

Figure 3.- Reduced system.

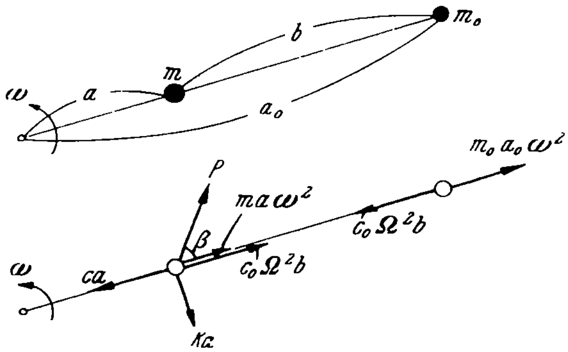

Figure 5.- Force diagram of substitute system.

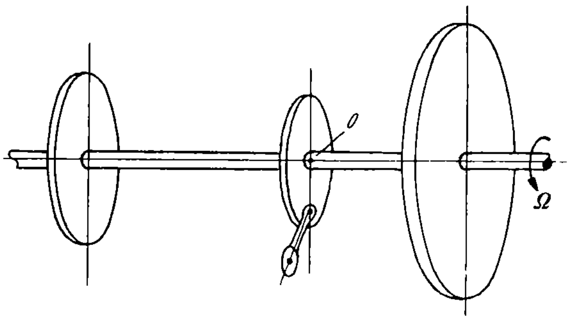

Figure 9.- Some vibratory system with a pendulum mass.

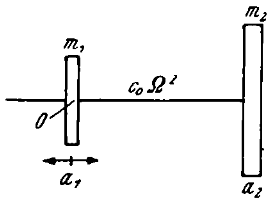

Figure 10.- Substitute system for the pendulum mass of Figure 9.

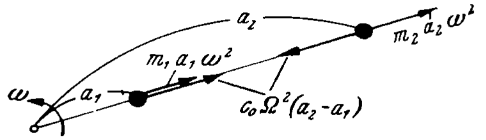

Figure 11.- Force diagram of the pendulum mass.

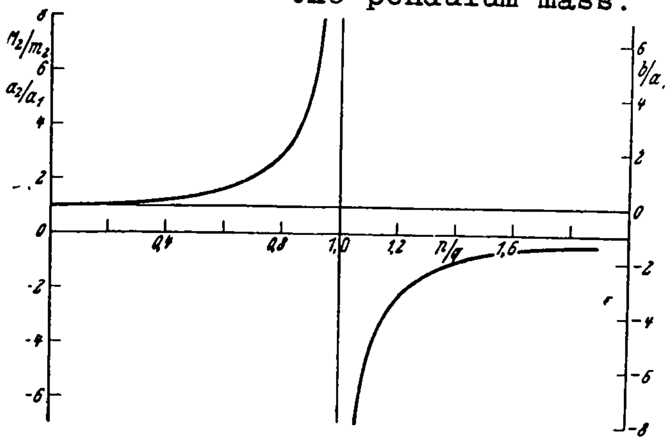

Figure 12.- Substitute mass and deflections by different tuning of pendulum mass.

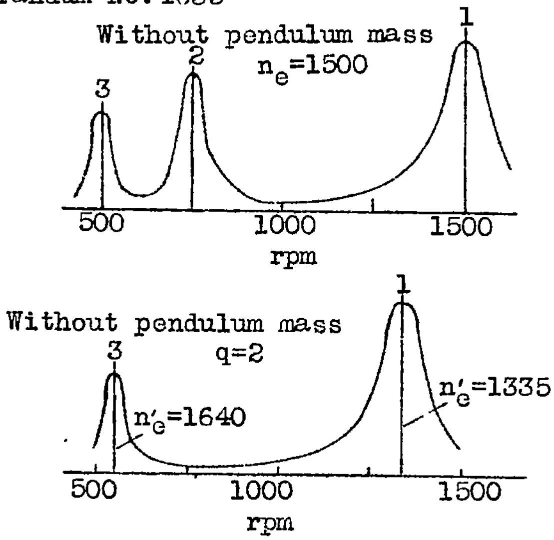

Figure 8.- Resonance curves of a single cylinder power plant with and without pendulum mass.

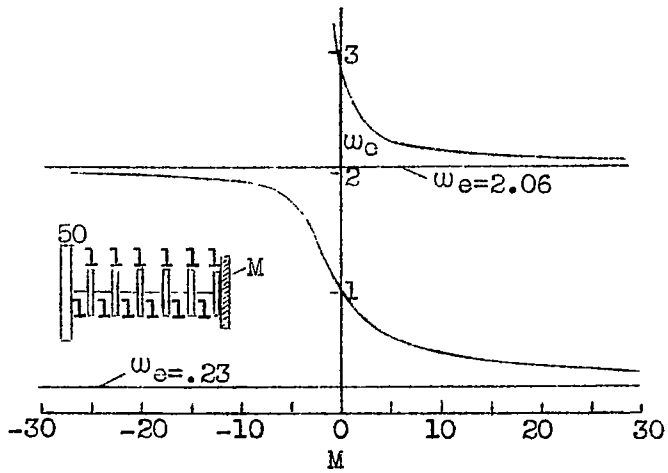

Figure 13.- Natural frequency of a multi-mass system for different values of substitute mass M.

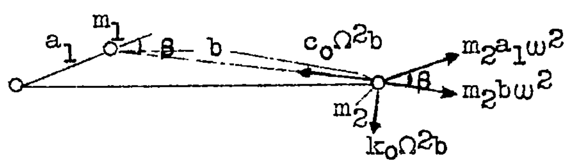

Figure 24.- Diagram of the forces of the pendulum mass by small natural damping.

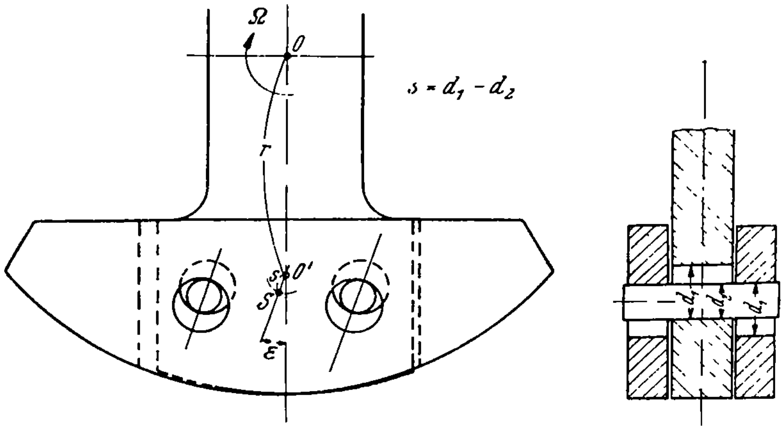

Figure 14.- Pendulum, Wright design version.

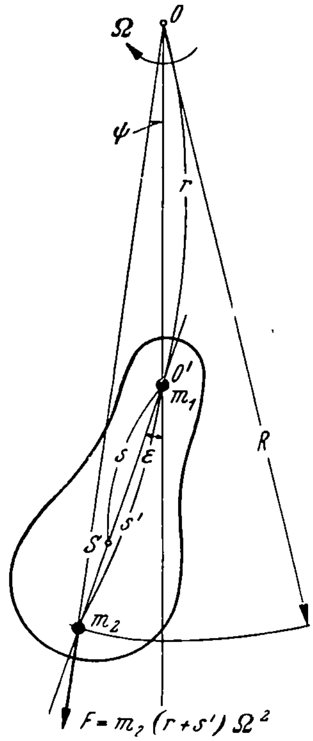

Figure 15.- Material pendulum.

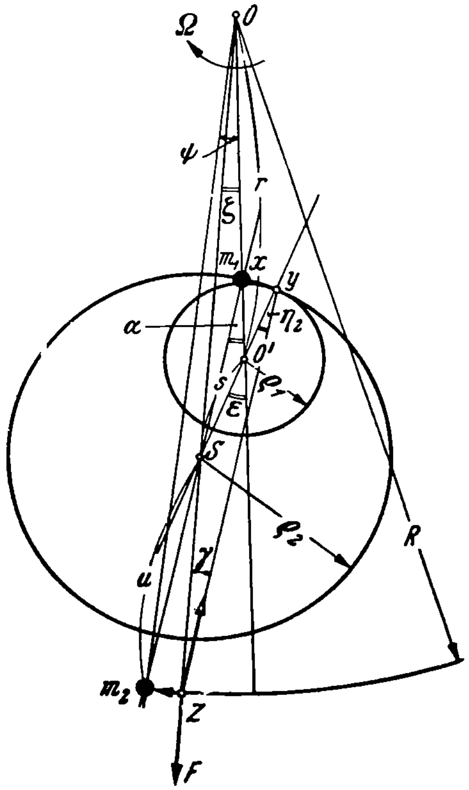

Figure 17.- Geometrical conditions on Salomon's outer roller.

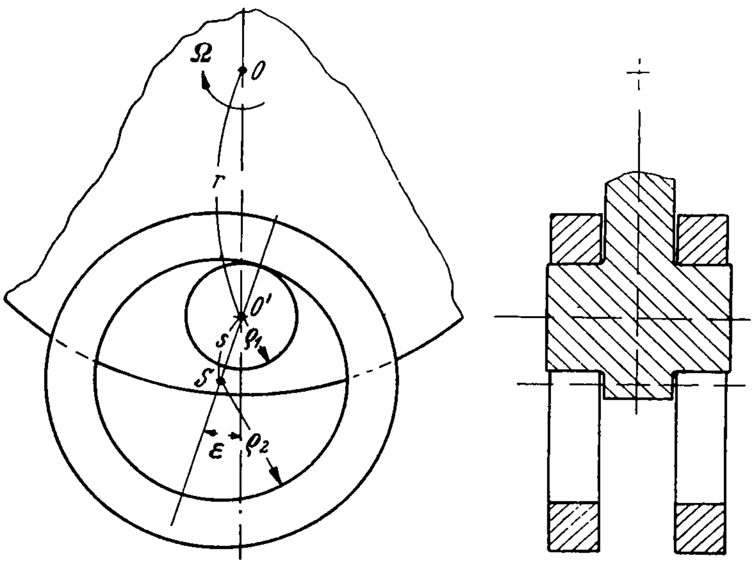

Figure 16.- Salomon's outer roller.

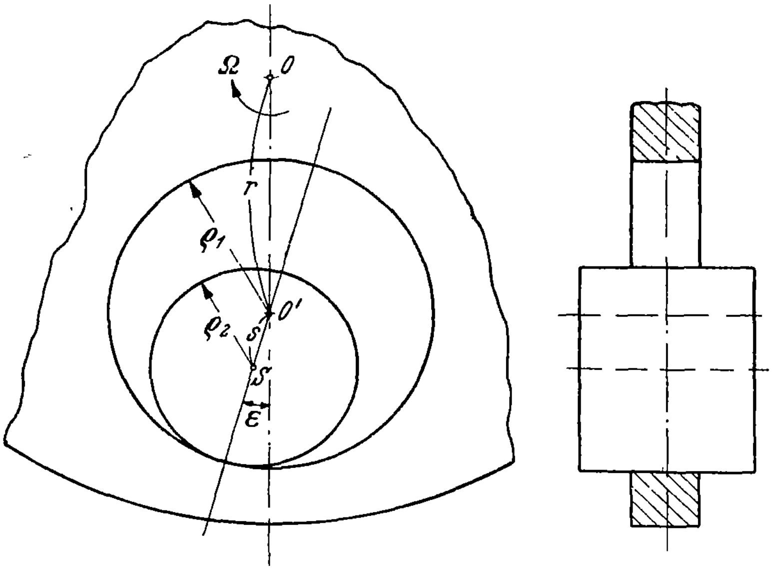

Figure 18.- Salomon's inside roller.

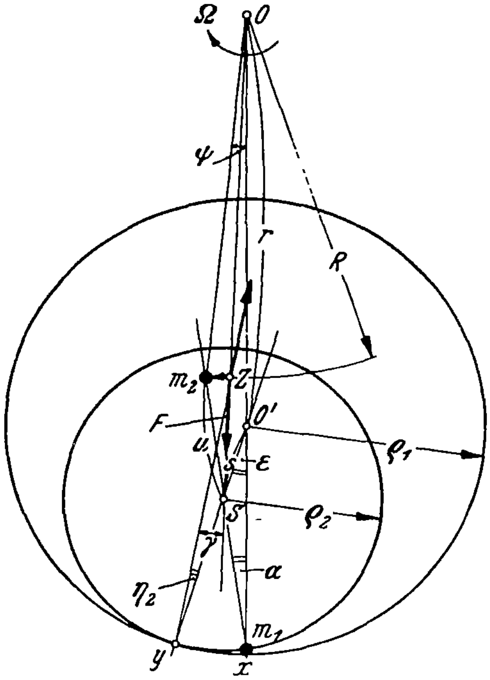

Figure 19.- Geometrical conditions on Salomon's inside roller.

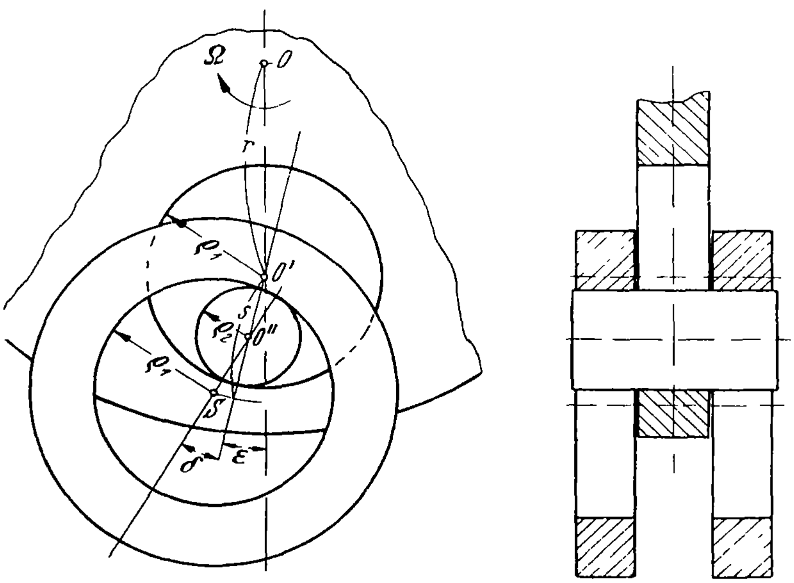

Figure 20.- Pendulum mass with two degrees of freedom.

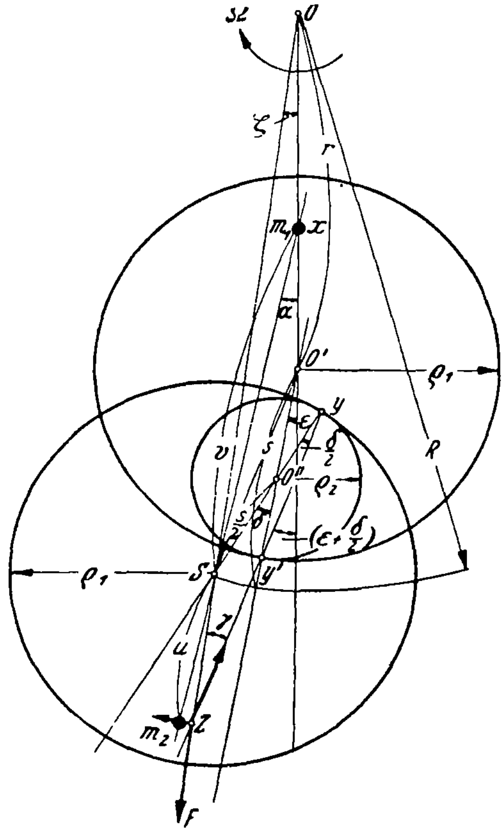

Figure 21.- Geometrical conditions of the pendulum mass with two degrees of freedom.

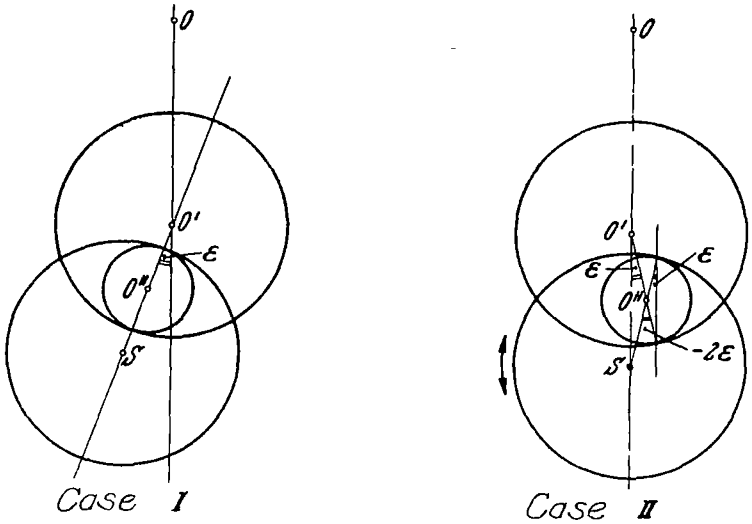

Figure 22.- The two possible natural vibrations of a pendulum mass having two degrees of freedom.

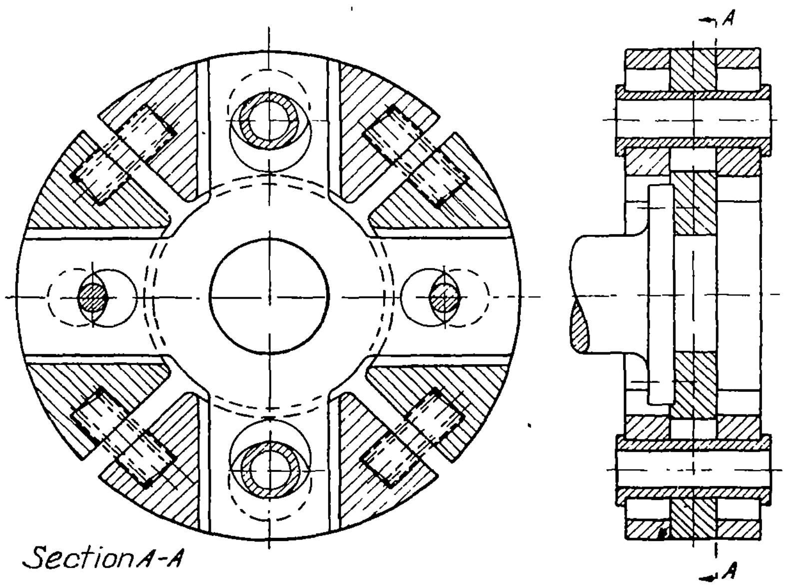

Figure 23.- Device with several pendulum masses of the type of Figure 20.